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Constructing the axial stiffness of longitudinally vibrating rod from fundamental mode shape

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Abstract

It is shown that inhomogeneous rods of uniform cross-section with the same material density variation and the same length, but with different boundary conditions and different elastic modulus variations along the axes of the rods have coincident fundamental natural frequency. The rods have different mode shapes that stem from the static displacements of the associated uniform rods, under uniformly distributed loads. These interesting properties appear in combination with other unanticipated results; namely, the natural frequency expression for any polynomial variation of the material density can be expressed in a unified manner, and depends solely on two coefficients: one coefficient describing material density and the other associated with elastic modulus variation.

Keywords: Vibrating rod; Mode shape; Axial stiffness

1. Introduction

Free longitudinal vibration of uniform, homogeneous rods is covered nearly in all vibration texts. The reader may consult with books by Weaver et al. (1990) and Rao (1995). Vibration frequencies of tapered rods were studied by Conway et al. (1964). Graf (1975) pointed out that for bars of conical cross-sections, the equation of motion could be put in the form of the wave equation by an appropriate change of variable. This idea was further developed by Abrate (1995) and Horgan and Chan (1999). Graf (1975) considered rods with various profiles, namely those designated as follows:

$$A(\xi) = A_0\xi \quad \text{(linear)}, \tag{1}$$

$$A(\xi) = A_0\xi^2 \quad \text{(conical)}, \tag{2}$$

$$A(\xi) = A_0 e^{\alpha\xi} \quad \text{(exponential)}, \tag{3}$$

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and

$$A(\xi) = A_0 \cosh^2(\beta\xi) \quad \text{(catenoidal)}, \tag{4}$$

where A_0 is the cross-sectional area at the origin of the coordinates, $\xi = x/L$ is the nondimensional axial coordinate, L is the length of the rod, α and β are positive constants.

For the rod with the linear area variation, Eq. (1), Graf (1975) obtained an exact solution in terms of the zeroth-order Bessel function, whereas for the conical rod, Eq. (2), the exact solution is obtainable via application of the half-order Bessel functions. Series solutions for the nonuniform bars were constructed by Eisenberger (1991a). In another paper of Eisenberger (1991b), "the exact solution is obtained using one element for each segment with continuously varying properties, and the displacements and stresses are exact all along this rod".

The present paper deals with closed-form solutions for uniform cross-section nonhomogeneous rods. Both the material density and the elastic modulus are treated as functions of the axial coordinate. Apparently, for the first time, closed-form solutions are presented for the nonhomogeneous rod under two sets of boundary conditions. Closed-form expressions for the natural frequency can serve as benchmark solutions.

2. Formulation of the problem

The differential equation governing the free longitudinal vibration of rods reads

$$\frac{\partial}{\partial x}\left[E(x)A(x)\frac{\partial u}{\partial x}\right] = \rho(x)A(x)\frac{\partial^2 u}{\partial t^2}, \tag{5}$$

where $u(x,t)$ is an axial displacement, i.e. a function of the axial coordinate x and time t; $E(x)$ is the modulus of elasticity i.e. varying along the axis, $\rho(x)$ and $A(x)$ are material density and the cross-sectional area, respectively, that are two functions of the axial coordinate x.

To find the natural frequency, we represent the displacement in the form

$$u(x,t) = U(x)e^{i\omega t}, \tag{6}$$

where $U(x)$ is the mode shape and ω is the natural frequency. Substituting Eq. (6) into Eq. (5) leads to

$$\frac{\partial}{\partial x}\left[E(x)A(x)\frac{\partial U}{\partial x}\right] + \rho(x)A(x)\omega^2 U = 0. \tag{7}$$

Consider rods whose ends are either clamped or free. At the clamped end, the boundary condition reads

$$U = 0, \tag{8}$$

whereas at the free end, the boundary condition is

$$dU/dx = 0. \tag{9}$$

For the uniform and homogeneous rod, $A(x) = \text{constant}$ and $E(x) = \text{constant}$ and $\rho(x) = \text{constant}$. We first find, as an auxiliary problem, the static displacements of such rods due to uniform axial loading. The static displacement for the clamped–clamped uniform rod reads (from the left-hand side of Eq. (5)):

$$U_{\text{C–C}}(\xi) = \xi - \xi^2, \tag{10}$$

where ξ is the nondimensional axial coordinate,

$$\xi = x/L. \tag{11}$$

For the clamped–free uniform rod, the associated displacement is

$$U_{\mathrm{C\text{-}F}}(\xi) = \xi - \xi^2/2. \tag{12}$$

We pose the following question: Are there nonuniform and/or nonhomogeneous unloaded rods such that expressions given in either Eq. (10) or Eq. (12) constitute the *exact* solution for the natural frequency? This question may appear to be an artificial one in the first place. Indeed, why should static displacement of the uniform and homogeneous rod serve as a mode shape of either a nonuniform or a nonhomogeneous rod?

In posing this question, we are guided here by the previous experience, derived for the nonhomogeneous columns and beams (Elishakoff and Rollot, 1999; Candan and Elishakoff, 2000), where indeed the above phenomenon took place for four different sets of boundary conditions. We must immediately remark that if there are rods whose mode shapes are given by Eq. (10) or Eq. (12), then these mode shapes correspond to the fundamental frequencies since the mode shapes have no internal nodes.

3. Inhomogeneous rods with uniform density

Consider first a clamped–clamped rod that has a constant material density $\rho(x) = a_0 > 0$, and a constant cross-sectional area A = constant. We ask the following question: What are the coefficients b_0, b_1 and b_2 in the expression for the modulus of elasticity

$$E(\xi) = b_0 + b_1\xi + b_2\xi^2, \tag{13}$$

so that the rod possesses a vibration mode given by Eq. (10)?

To this end, we substitute Eqs. (10) and (13) into the differential equation (7). The result is

$$b_1 - 2b_0 + (-4b_1 + 2b_2 + a_0k)\xi, \quad -(6b_2 + a_0k)\xi^2 = 0, \tag{14}$$

where

$$k = \omega^2 L^4. \tag{15}$$

Since this expression must be valid for each ξ, we conclude that

$$b_1 - 2b_0 = 0, \tag{16}$$

$$-4b_1 + 2b_2 + a_0k = 0, \tag{17}$$

$$6b_2 + a_0k = 0. \tag{18}$$

From Eqs. (16)–(18), we have three equations for the four unknowns b_0, b_1, b_2 and k. Setting b_2 arbitrary, we solve for b_0, b_1 and k in terms of b_2 and obtain

$$b_0 = -b_2, \tag{19}$$

$$b_1 = -b_2/2, \tag{20}$$

$$k = -6b_2/a_0. \tag{21}$$

Thus, if the modulus of elasticity of the inhomogeneous rod reads

$$E(\xi) = -1/2b_2 - b_2\xi + b_2\xi^2, \tag{22}$$

then the natural frequency squared is given by

$$\omega^2_{\text{C–C}} = -6b_2/L^2 a_0 \tag{23}$$

with the mode shape given in Eq. (10). Since a_0 is the density of the rod, it has to be positive. Therefore, in order for the resulting expression for ω^2 to be positive, b_2 must be negative. Note that $E(\xi)$ must be positive throughout the interval [0,1]. It is easy to confirm that $E(\xi)$ in Eq. (22) is positive. Since b_2 is negative, we have that the trinomial $E(\xi)/|b_2| = 1/2 + \xi - \xi^2$ must be positive in [0,1]. It is positive when $\xi = 0$, and has the positive root $(1 + \sqrt{3})/2$, which is greater than one. Thus, $E(\xi)$ is positive in [0,1].

Consider now the nonhomogeneous clamped–free rod of uniform density and modulus of elasticity given in Eq. (13). The mode shape in these new circumstances is given in Eq. (12). We substitute Eqs. (12) and (13) into the governing differential equation (7). The result is a polynomial equation

$$(b_1 - b_0) + (-2b_1 + 2b_2 + ka_0)\xi - (3b_2 - \tfrac{1}{2}ka_0)\xi^2 = 0. \tag{24}$$

Hence, the expressions in the parentheses must vanish. The resulting equations can be put in the following convenient form

$$b_1 - b_0 = 0, \tag{25}$$

$$-2b_1 + 2b_2 + ka_0 = 0, \tag{26}$$

$$3b_2 + \tfrac{1}{2}ka_0 = 0. \tag{27}$$

With b_2 as an arbitrary parameter, we solve for b_0, b_1 and k in terms of b_2 and obtain

$$b_0 = -2b_2, \qquad b_1 = -2b_2, \qquad k = -6b_2/a_0. \tag{28}$$

We arrive at the following conclusion: if the modulus of elasticity is given by

$$E(\xi) = -2b_2 - 2b_2\xi + b_2\xi^2, \tag{29}$$

then the mode shape is given in Eq. (12), and the natural frequency squared is given by the following closed-form expression

$$\omega^2_{\text{C–F}} = -6b_2/L^2 a_0. \tag{30}$$

Here too, the value b_2 is negative; $E(\xi)$ in Eq. (29) is positive throughout the interval $0 \leqslant \xi \leqslant 1$.

We also observe that the fundamental frequency expressions for the *clamped–clamped* (C–C) rod in Eq. (23) and for the *clamped–free* (C-F) rod in Eq. (30) coincide if the coefficient in front of ξ^2 in $E(\xi)$ is the same in both cases. At first glance this may appear to be a paradoxical situation, since these two rods possess *different* mode shapes and different elastic moduli. To explain this result, let us list the fundamental frequencies of the associated uniform rods of the same length L:

$$\omega_{\text{C–C}} = \frac{\pi}{L}\sqrt{\frac{E_1}{\rho_1}}, \qquad \omega_{\text{C-F}} = \frac{\pi}{2L}\sqrt{\frac{E_2}{\rho_2}}, \tag{31}$$

where ρ_1 and E_1 are the material density and modulus of elasticity of the clamped–clamped rod, whereas ρ_2 and E_2 are the material density and modulus of elasticity of the clamped–free rod, respectively. If the material densities are equal $\rho_1 = \rho_2 = a_0$, but $E_2 = 4E_1$, then the fundamental natural frequencies coincide. This implies that two rods can share the same fundamental frequencies if their material characteristics differ. As discussed above, when elastic moduli of C–C and C-F rods differ they may share the same natural frequency. In the case of the inhomogeneous rods, considered above, the expressions for the elastic moduli in Eqs. (22) and (29) differ. Thus, the coincidence of frequencies should not be totally unexpected. Yet, let us review the conditions that lead to the coincidence of the natural frequencies. We *postulated* the vibration modes and looked for rods with *polynomial* variation of density and elastic modulus that possess the given mode shape. Such a search, remarkably, led to the same fundamental frequency of rods in two sets of boundary conditions. It will be shown below that this interesting phenomenon repeats itself for rods with nonconstant densities along the rod's axis.

4. Inhomogeneous rods with linearly varying density

Consider rods whose material density is represented by the following function:

$$\rho(\xi) = a_0 + a_1\xi. \tag{32}$$

We are looking for a rod with cubic polynomial representing its modulus of elasticity variation

$$E(\xi) = b_0 + b_1\xi + b_2\xi^2 + b_3\xi^3. \tag{33}$$

Note that the highest order in the polynomial expression for $E(\xi)$ is three, whereas in $\rho(\xi)$ it is unity. This is due to the fact that two differentiations are involved in the first term of the differential equation (7).

Substitution of Eqs. (32) and (33) into the governing equation, in view of the mode shape for the clamped–clamped rod (10) results in the polynomial expression as the left-hand side of the equation. Since it is valid for any ξ, we get the following set of equations:

$$\begin{aligned} b_1 - 2b_0 &= 0, \\ -4b_1 + 2b_2 + a_0k &= 0, \\ -6b_2 + 3b_3 - a_0k + a_1k &= 0, \\ -8b_3 - a_1k &= 0. \end{aligned} \tag{34}$$

We have four equations for five unknowns: b_0, b_1, b_2, b_3 and k. We express unknowns in terms of b_3, as follows:

$$\begin{aligned} b_0 &= b_3(5a_1 + 16a_0)/24a_1, \\ b_1 &= -b_3(5a_1 + 16a_0)/12a_1, \\ b_2 &= b_3(-5a_1 + 8a_0)/6a_1. \end{aligned} \tag{35}$$

We arrive at the following result: if the material density is given in Eq. (31), and the elastic modulus variation is

$$E(\xi) = \left[-\left(\frac{5}{3}+\frac{8}{3}\frac{a_0}{a_1}\right)+\left(\frac{5}{3}+\frac{8}{3}\frac{a_0}{a_1}\right)\xi-\left(-\frac{5}{3}+\frac{4}{3}\frac{a_0}{a_1}\right)\xi^2+\xi^3\right]b_3, \tag{36}$$

then the natural frequency of the clamped–clamped rod is given by the last expression in Eq. (34)

$$\omega^2_{\mathrm{C\text{–}C}} = -8b_3/a_1L^2. \tag{37}$$

For the clamped–free rod, expressions (32) and (33) should be utilized in conjunction with the mode shape in Eq. (12), to be substituted into the differential equation (7). It is valid if the following conditions are met:

$$\begin{aligned} b_1 - 2b_0 &= 0, \\ -2b_1 + 2b_2 + ka_0 &= 0, \\ -3b_2 + 3b_3 - ka_0/2 + ka_1 &= 0, \\ -4b_3 - ka_1/2 &= 0. \end{aligned} \tag{38}$$

With b_3 as parameter, solving for b_0, b_1, b_2 and k in terms of b_3 we obtain

$$\begin{aligned} b_0 &= \left(-\frac{5}{3}+\frac{8}{3}\frac{a_0}{a_1}\right)b_3, \\ b_1 &= \left(-\frac{5}{3}+\frac{8}{3}\frac{a_0}{a_1}\right)b_3, \\ b_2 &= \left(-\frac{5}{3}+\frac{4}{3}\frac{a_0}{a_1}\right)b_3, \\ k &= -8\frac{b_3}{a_1}. \end{aligned} \tag{39}$$

Solution of the above equations leads to the coefficients b_0, b_1, b_2 and b_3. The final expression for the modulus of elasticity is

$$E(\xi) = \left[\left(-\frac{5}{3}+\frac{8}{3}\frac{a_0}{a_1}\right)+\left(-\frac{5}{3}+\frac{8}{3}\frac{a_0}{a_1}\right)\xi+\left(-\frac{5}{3}+\frac{4}{3}\frac{a_0}{a_1}\right)\xi^2+\xi^3\right]b_3. \tag{40}$$

Thus, the inhomogeneous clamped–free rod with linearly-varying material density and variable modulus of elasticity in Eq. (40) has the fundamental natural frequency stemming from the last expression in Eq. (39):

$$\omega^2_{\mathrm{C\text{–}F}} = -8b_3a_2/L^2. \tag{41}$$

5. Rod with general variation of material density ($m > 2$)

Now consider the general case, $m > 2$. We are looking for the following variations in the density and elastic modulus variations:

$$\begin{aligned} \rho(\xi) &= \sum_{i=0}^{m} a_i\xi^i, \\ E(\xi) &= \sum_{i=0}^{m+2} b_i\xi^i. \end{aligned} \tag{42}$$

For the clamped–clamped rod, we are looking for the rods that possess the mode shape given in Eq. (10). Substitution into Eq. (5) yields

$$\sum_{i=0}^{m+2}(i+1)b_{i+1}\xi^i - 2\sum_{i=1}^{m+2}b_i\xi^i(i+1) - 2b_0 + k\sum_{i=1}^{m+1}a_{i-1}\xi^i - k\sum_{i=2}^{m+2}a_{i-2}\xi^i = 0, \tag{43}$$

$$b_1 - b_0 = 0, \tag{44}$$

$$\begin{aligned} &2b_2 - 4b_1 + ka_0 = 0, \\ &\vdots \end{aligned} \tag{45}$$

$$\begin{aligned} &(i+1)b_{i+1} - 2b_i(i+1) + ka_{i-1} - ka_{i-2} = 0 \qquad (2 \leqslant i \leqslant m+1), \\ &\vdots \end{aligned} \tag{46}$$

$$-2(m+3)b_{m+2} - ka_m = 0. \tag{47}$$

We obtain

$$\begin{aligned}
b_0 &= b_1/2,\\
\omega^2 &= (4b_1 - 2b_2)/L^2 a_0,\\
&\vdots\\
\omega^2 &= \frac{(i+1)(2b_i - b_{i+1})}{L^2(a_{i-1} - a_{i-2})},\\
&\vdots\\
\omega^2 &= -2(m+3)b_{m+2}/L^2 a_m.
\end{aligned} \tag{48}$$

For the coefficients b_i, we obtain

$$\begin{aligned}
b_0 &= b_1/2,\\
b_1 &= [2(2a_0 + a_1)b_2 - 3a_0 b_3]/4(a_1 - a_0),\\
&\vdots\\
b_i &= \{[i(a_i - a_{i-2} - 2a_{i-1}) + a_1 + 3a_{i-1} - 4a_{i-2}]b_{i+1}\\
&\quad + (i+2)(a_{i-2} - a_{i-1})b_{i+2}\}/[2(i+1)(a_i - a_{i-1})],\\
&\vdots\\
b_{m+1} &= \frac{b_{m+2}[-(2m+8)a_m + 2(m+3)a_{m-1}]}{2a_m(m+2)}.
\end{aligned} \tag{49}$$

As for the clamped–free rod, the result of substitution of Eqs. (42) into Eq. (5) in conjunction with Eq. (12) leads to

$$\sum_{i=0}^{m+1}(i+1)b_{i+1}\xi^i - \sum_{i=1}^{m+2}b_i(i+1)\xi^i - b_0 + k\sum_{i=1}^{m+1}a_{i-1}\xi^i - k\sum_{i=2}^{m+2}a_{i-2}\xi^i = 0. \tag{50}$$

Since this equation must be valid for any ξ, we arrive at the following recurrent equations:

$$\begin{aligned}
b_1 - b_0 &= 0,\\
2b_2 - 2b_1 + ka_0 &= 0,\\
&\vdots\\
(i+1)b_{i+1} - b_i(i+1) + ka_{i-1} - ka_{i-2}/2 &= 0, \quad (2 \leqslant i \leqslant m+1),\\
-(m+3)b_{m+2} - ka_m/2 &= 0.
\end{aligned} \tag{51}$$

These equations result in

$$
\begin{aligned}
&b_0 = b_1,\\
&\omega^2 = 2(b_1 - b_0)/L^2 a_0,\\
&\quad\vdots\\
&\omega^2 = \frac{(i+1)(b_1 - b_{i+1})}{L^2(a_{i-1} - a_{i-2}/L)},\\
&\quad\vdots\\
&\omega^2 = -2(m+3)b_{m+2}/L^2 a_m.
\end{aligned}
\tag{52}
$$

Note that the expression for ω^2 in Eq. (52) is the same as in Eq. (48). Compatibility of Eq. (52) yields

$$
\begin{aligned}
&b_0 = b_1,\\
&b_1 = [2(a_1 + a_0)b_2 - 3a_0 b_3]/(a_0 - 2a_1),\\
&\quad\vdots\\
&b_i = \frac{1}{(i+1)(2a_i - a_{i-1})}\{[i(a_{i-1} - a_{i-2} + 2a_2)] + [3a_{i-1} - 2a_{i-2} + 2a_i]b_{m+1}\\
&\qquad + [i(a_{i-1} - 2a_{i-1}) + 2(a_{i-2} - 2a_{i-3})]b_{i+2}\},\\
&\quad\vdots\\
&b_{m+1} = \frac{[(m+3)a_{m-2} - (m+4)a_m]}{a_m(m+2)} b_{m+2}.
\end{aligned}
\tag{53}
$$

The final expression for the natural frequency squared is

$$\omega^2 = -2(m+3)b_{m+2}/L^2 a_m. \tag{54}$$

In order for the natural frequency to be a positive quantity, it is necessary that a_m and b_{m+2} have opposite signs.

6. Discussion

It should be emphasized that all previous expressions of the natural frequency can be put in the form (54) with proper choice of m. However, the expressions for b_i are derivable separately, for $m = 0$, 1, 2. To the best of the authors' knowledge, most of the previous investigators considered nonuniform rods in which the cross-section varied but the material properties remained constant. It can then be argued that one can actually manufacture such a rod so that the problem being studied can have direct applications. The present paper considers cases in which the cross-sectional area is constant and the modulus of elasticity and/or the density of the material vary continuously with position. This raises the most important question about this study: Is it addressing a problem of practical interest, or is this simply an academic exercise in finding solutions to a differential equation? How does one make a rod with properties varying as prescribed in this study? To answer these somewhat provocative questions, we first note that early studies in inverse problems did not exclude variable material properties. For example, Krein (1952) considered a string with variable material density. More recently, Ram and Elhay (1998) distinguished between the cases, where A and E are constant, while ρ varies with ξ, or A, E and ρ all vary with ξ. Important variables are, of course, the products EA and ρA. As far as the manufacturing of the rods with given axial variation of $E(\xi)$ and $\rho(\xi)$ is concerned, even if such a procedure does not exist today, its development in the future cannot be a priori excluded.

The coincidence of the *fundamental* natural frequencies of rods with different boundary conditions may appear to be a surprising fact, at the first glance. We have addressed this question above, albeit briefly. It is remarkable that there exist structures that have the same *complete* spectrum. For example, Gottlieb and McManus (1998) illustrated how two different polygonal membranes may have the same *entire* spectrum, thus forming the so called isospectral structures. The reader may also consult the work by Gottlieb (1989), Gladwell and Morassi (1995), Chapman (1995), and Sridhar and Kudrolli (1994) who experimentally verified the isospectral property of membranes of different shapes.

It is noteworthy to remark on the similarity and difference of the present study with the general topic of inverse problems, covered, for example, in the definitive monographs by Gladwell (1986) and Tarantolla (1987). As Tarantolla (1987) writes, "to solve the *forward problem* is to predict the values of the observable parameters, given arbitrary values of the model parameters. To solve the *inverse problem* is to infer the values of the model parameters from the given observed values of the observable parameters" (italics by Tarantolla). In vibration context, the inverse problem consists in reconstructing the structure by its observable vibration spectrum. The reconstruction of the continuous variations in axial rigidity and the material density of a longitudinally vibrating rod was studied apparently independently by Ram (1994) and Wang and Wang (1994). In particular, Ram (1994) proved that the density and axial rigidity functions are uniquely determined by two natural frequencies, their corresponding mode shapes, and the total mass of the rod, when specially derived necessary and sufficient conditions are met for the construction of the physically realizable rod, i.e. with positive parameters. Wang and Wang (1994) demonstrated that for the rod's reconstruction one needs the knowledge of two positive square frequencies, two associated mode functions with piece-wise continuous second-order derivatives, satisfying some necessary conditions. The objective of the present work is *different* from those of Ram (1994) or Wang and Wang (1994) (see also Gladwell and Gbadeyan, 1985 and Ueda, S., 1988). Here, we are looking for closed-form solutions for natural frequencies with specified fundamental vibration *mode* alone. In these circumstances, we uncovered an infinite *number* of closed-form solutions, corresponding to the degree of variation in the mass density, with $m = 0, 1, 2, \ldots$

In Ram's terminology, "in the classical inverse problem, it is assumed that the cross-sectional area of the rod is variable, while Young's modulus of elasticity and the rod density are constants." Studies by Ram (1994) and Wang and Wang (1994) allow the cross-sectional area, the modulus of elasticity and the material density to *vary* along the rod's axial coordinate. The present work is devoted to rods with the cross-sectional area as a constant. Even with this seeming restriction, an *infinite* number of rods are uncovered that possess a given polynomial mode shape. Once technology exists that allows construction of inhomogeneous rods with polynomially varying modulus of elasticity, it is an easy task to demand the rod to have any preselected fundamental natural frequency. Indeed, as Eq. (53) indicates the fundamental frequency depends solely upon a_m and b_{m+2}. If a technology allows for manufacturing rods with arbitrary a_m and b_{m+2}, one can get any desirable fundamental natural frequency. This leads both to an *avoiding resonance* condition for forced deterministic vibration, and the first frequency to lie outside the range of excitation of a rod under cutoff white noise with two cutoff frequencies $\omega_{C,1}$ and $\omega_{C,2}$ in the random vibration environment (Elishakoff, 1999). If the fundamental frequency ω_1 in less than $\omega_{C,1}$, the response level can be significantly reduced. Thus, a new passive vibration control mechanism may be obtained.

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